Discrete Math Midterm

Ashish Jayamohan - G08141951
Start Time: 1:00
End Time: 2:17

Question 1

We are asked to prove that for integers a, b, and c, $a|c$ where $a|b$ and $a|(b+c)$.
Assuming $a|b$ and $a|(b+c)$ are both true, we can conclude that there exists some x and y where the following statements stand true:

$$x \cdot a = b$$

$$y \cdot a = b + c$$

Using systems of equations, we can subtract b from $b+c$, giving us the following equation:

$$(b+c) - b = c = (y \cdot a) - (x \cdot a)$$

Factoring on the right hand side gives us the following:

$$c = a \cdot (y - x)$$

Thus, since x and y are both integers, their difference is also an integer, signifying that there is some integer $y - x$ which multiplies by a to give c, proving that $a|c$

Question 2

We are asked to prove that for a given integer n, $n^2 - 2$ is not divisible by 4.
In the case where $n^2 - 2$ is divisible by 4, we can assume the following statement is true as well for some arbitrary x where $x \in \mathbb{Z}$:

$$n^2 - 2 = 4 \cdot x$$

$$n^2 = 4x + 2$$

We now have to consider two cases - one where n is even and the other where n is odd.
Where n is even, we can assume that $n = 2 \cdot y$ where y is some arbitrary number where $y \in \mathbb{Z}$.
Knowing this, we can use our statement from above and derive the following:

$$(2y)^2 = 4 \cdot 4x + 2 \rightarrow 4y^2 = 4x + 2$$

We can now define some arbitrary value a such that $a = y^2$ and $a \in \mathbb{Z}$. Using this, we can substitute to find the following:

$$4a = 4x + 2$$

Dividing by 2 on both sides, we produce:

$$2s = 2x + 1$$

We know that $2s$ must be even and $2x + 1$ must be odd. Thus, because an even number cannot also be odd, we have a contradiction.
We also consider the case where n is odd. In this case, we can define some variable z such that $(2z + 1)^2 = 4x + 2$ and $z \in \mathbb{Z}$. Simplifying this statement, we produce:

$$4z^2 + 4z + 1 = 4x + 2$$

$$4z^2 + 4z = 4x + 1$$

Factoring the 4 out of the left-hand side of the equation, we get:

$$4(z^2 + z) = 4x + 1$$

Assigning b as some number where $b \in \mathbb{Z}$ and $b = z^2 + z$, we produce the following equation:

$$4b = 4x + 1$$

We can now define some number c as $c = 2x$ where $c \in \mathbb{Z}$. This produces the following when substituted:

$$2b = 2c + 1$$

Once again, $2b$ is even which $2c + 1$ is odd, thus proving a contradiction.
All cases possible for n prove to be a contradiction, thus we can conclude that $n^2 - 2$ is not divisible by 4 for any integer n.

Question 3

Part A

> Knowing n to be an integer, $n \mod 3$ can produce either 0, 1, or 2.

Part B

> An integer n can be written in the following ways assuming a to be some arbitrary number where $a \in \mathbb{Z}$:
>
> $$3a$$
>
> $$3a + 1$$
>
> $$3a + 2$$

Part C

Using our intuition from Part B, we can presume that there are three remainders r which we can get from the calculation $r = n \mod 3$ for some number n where $n \in \mathbb{Z}$. The three values which r can take on are 0, 1, and 2. Thus, we can represent every integer n as either $3a$, $3a + 1$, or $3a + 2$ for some integer a where $a \in \mathbb{Z}$ as well. Thus, we have three cases to consider for this problem.
In the case where we can represent our n as $3a$, we know that n itself is composite as it is divisible by 3. The only possible loop in this would be when $n = 3$, but we are told in the problem that $n > 3$, thus invalidating this exception.
In the case where have an n which is represented as $3a + 1$, $n + 2$ can be represented as such:

$$n + 2 = 3a + 3$$

We can factor the 3 out of the right hand side of the equation to reveal the following:

$$n + 2 = 3 \cdot (a + 1)$$

Thus, we can see that in this case, $n + 2$ is divisible by 3, thus making it a composite number.
Finally, in the case where n is represented as $3a + 2$, $n + 4$ can be represented as such:

$$n + 4 = 3a + 6$$

Factoring out 3 on the right hand side once again, we produce the following:

$$n + 4 = 3 \cdot (a + 2)$$

Thus, $n + 4$ in this case is non-prime as it is divisible by 3.
Thus, for all cases of n, either n, $n + 2$, or $n + 4$ is a composite number.

Question 4

We are asked to prove that $1 + 3n \leq 4^n$ where $n \geq 0$.
Let us define $\theta(n)$ as representing the inequality above. As a simple case, we find that $\theta(0)$ is true.
Assuming k to be some integer where $k \geq 0$ where $\theta(k)$ is true, we can establish the following inequality:

$$1 + 3k \leq 4^k$$

By induction, we increment k by 1 will give us the following inequality:

$$1 + 3(k + 1) \leq 4^{k+1}$$

Simplifying and redistributing the right side of the inequality, we obtain: $(3k + 1) + 3$. By induction, this must be less than or equal to $4^n + 3 \leq 4^n + 3 \cdot 4^n$. Thus, $4^n + 3 \leq 4^{n+1}$, proving that $\theta(n + 1)$ is true. Thus, for any n where $n \in \mathbb{Z}$ and $n \geq 0$, $1 + 3n \leq 4^n$.