Discrete Math Homework 2

4.6.18

We can assume that both a and d are integers where $d > 0$ and such integers q_1, q_2, r_1, and r_2 exist such that the following conditions are fulfilled:

$$d \cdot q_1 + r_1 = d \cdot q_2 + r_2$$

Assuming this to be true, we can then use simple algebra to derive:

$$r_2 - r_1 = d \cdot (q_1 - q_2)$$

We can be certain that both r_1 and r_2 fall between 0 and d as it signifies the remainder. As a result, we can assume the following conditions to be true:

$$0 \leq r_1 < d$$

$$0 \leq r_2 < d$$

Using these assumptions, we can derive the following:

$$-d < r_2 - r_1 < d$$

Knowing that $r_2 - r_1 = d \cdot (q_1 - q_2)$, we can substitute to get the following:

$$-d < d \cdot (q_1 - q_2) < d$$

Knowing that $d > 0$, we can divide the entire inequality by d, producing the following:

$$-1 < q_1 - q_2 < 1$$

Since we know that $q_1 - q_2$ must be an integer ($a - b \in \mathbb{Z}$ where $a, b \in \mathbb{Z}$), we can conclude that the only possible value for $q_1 - q_2$ is 0. Thus, we can conclude that since $q_1 - q_2 = 0$, $q_1 = q_2$. Because we know that $q_1 = q_2$ and $r_2 - r_1 = d \cdot (q_1 - q_2)$, we can follow this stream of logic:

$$r_2 - r_1 = d \cdot 0$$

$$r_2 - r_1 = 0$$

$$r_1 = r_2$$

Thus, we conclude that $q_1 = q_2$ and $r_1 = r_2$

5.1.48

Since we are given that $i = k + 1$, we can simple substitute all k with $i - 1$. Knowing this, we can follow this stream of logic:

$$\sum_{k=1}^{5} k \cdot (k-1)$$

$$= \sum_{i=2}^{6} (i-1) \cdot ((i-1)-1)$$

$$= \sum_{i=2}^{6} (i-1) \cdot (i-2)$$

5.2.10

We are asked to prove the following where $n \geq 1$:

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

We begin assuming some integer m where $m = n$. Substituting, we get:

$$1^2 + 2^2 + \cdots + m^2 = \frac{m(m+1)(2m+1)}{6}$$

Thus, we can also conclude that the following is also true:

$$1^2 + 2^2 + \cdots + m^2 + (m+1)^2 = \frac{(m+1)(m+2)(2m+3)}{6}$$

We can also presume the following is true:

$$(1^2 + 2^2 + \cdots + m^2) + (m+1)^2 = \frac{m(m+1)(2m+1)}{6} + (m+1)^2$$

We can thus follow this stream of logic:

$$(1^2 + 2^2 + \cdots + m^2) + (m+1)^2 = \frac{m(m+1)(2m+1) + 6(m+1)^2}{6}$$

$$= \frac{(m+1) \cdot \{m(2m+1) + 6(m+1)\}}{6}$$

$$= \frac{(m+1)(2m^2 + 7m + 6)}{6}$$

$$= \frac{(m+1)(m+2)(2m+3)}{6}$$

Thus, because we arrive at this result which we predicted above, we have proven that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

5.3.29

Let us assume $P(n)$ where $P(n) = \frac{n(n-1)}{2}$. We can also assume an integer k such that $P(k) = \frac{k(k-1)}{2}$. If we assume that the prior equation is true for k people in the room, we can easily expand this situation to $k+1$ people in the room. With one extra person, that extra person would need to give handshakes to all the people existing in the room (k). Thus, we can say that the following is true:

$$P(k+1) = \frac{k(k-1)}{2} + k$$

$$= \frac{k^2 - k + 2k}{2}$$

$$= \frac{k^2 + k}{2} = \frac{k(k+1)}{2}$$

Using our initial formula of $P(n) = \frac{n(n-1)}{2}$, we can see that $P(k+1) = \frac{(k+1)k}{2}$, which is exactly what we have derive. Thus, the situation is proven.

5.4.7

For some number k, we can assume the following is correct, given the formula in the problem:

$$g_k - g_{k-1} = 2 \cdot (g_{k-1} - g_{k-2})$$

Extending this formula, we can extrapolate the following:

$$2 \cdot (g_{k-1} - g_{k-2}) = 2^{k-2} \cdot (g_2 - g_1) = 2^{k-1}$$

We can analyze this pattern as such:

$$g_2 - g_1 = 2$$

$$g_3 - g_2 = 4$$

$$\cdots$$

$$g_n - g_{n-1} = 2^{n-1}$$

We know that $g_n - g_1$ for some arbitrary n must be $2 + 4 + \cdots + 2^{n-1}$, which is equal to $2^n - 2$. Thus, we know that g_n must be $2^n - 2 + g_1$, which is equal to $2^n + 1$

5.5.38

Let us define c_n as the number of distinct ways in which to climb n stairs.
Where $n = 1$, $c_1 = 1$ and where $n = 2$, $c_2 = 2$. This is solved intuitively. We can therefore generalize the following where $n \geq 3$, there are two options: the last step being either 1 or 2 steps. In the case of a 1 step as the last step, there are c_{n-1} ways to reach the last step. Similarly, the case of 2 steps as the last move brings about c_{n-2} ways to reach the last set of 2

stairs. Thus, the number of ways that we can reach a certain n number of steps using the two aforementioned step sizes is by summing these two, generating the following:

$$c_n = c_{n-1} + c_{n-2}$$

. Thus, we can conclude the following for the situation:

$$c_1 = 1, c_2 = 2$$

$$c_n = c_{n-1} + c_{n-2} \,|\, n \geq 3$$

5.6.25

Let us assume that n is the input size for the program. Let us also assume that O_n signifies the number of operations done for a particular number of inputs, n.
We are given, in the problem that $O_1 = 7$ and that $O_n = O_{n-1} \cdot 2$. Knowing this, we can intuitively say that $O_n = 7 \cdot 2^{n-1}$. Knowing this, we can plug in 25 as our input to compute the answer:

$$O_{25} = 7 \cdot 2^{24}$$